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Isometries of Non-self-adjoint Operator Algebras

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We study isometries of certain non-self-adjoint operator algebras by means of the structure of the complete holomorphic vector fields on their unit balls and the associated partial Jordan triple products. We show that isometries of nest subalgebras of $B(H)$ are of the form $T \mapsto UTW$ or $T \mapsto UJT^*JW$, where U, W are suitable unitary operators and J a fixed involution of H. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper deals with isometries of certain non-self-adjoint algebras of operators on a Hilbert space. Our main theme is that the isometries must preserve besides the norm and the linear structure also the Jordan structure. In a way, this can be viewed as the generalization of the famous theorem of R. Kadison [K] (proving that every isometry of one C^*-algebra onto another is given by a Jordan isomorphism followed by a unitary multiplication) to the non-self-adjoint case.

Kadison used as the main tool the structure of the extreme points of the unit ball of a C^*-algebra (maximal partial isometries), i.e., the affine structure of the unit ball. Our approach is quite different and depends heavily on the holomorphic structure of the open unit ball. The class of algebras we consider is much bigger than C^*-algebras and includes nest algebras and unital Jordan subalgebras of JB^*-algebras.

If E is a complex Banach space with an open unit ball D, then the set $\mathrm{aut}(D)$ of all complete holomorphic vector fields on D is a real Banach Lie algebra. The group $\mathrm{Aut}(D)$ of all biholomorphic automorphisms of D is a real Banach Lie group, having $\mathrm{aut}(D)$ as its lie algebra. The *symmetric part* of E is $E_s = \mathrm{aut}(D)(0)$ and it is a closed, complex-linear subspace of E. The importance of this subspace stems from the fact that it is preserved by all members of $\mathrm{aut}(D)$, and its unit ball is invariant under all members of $\mathrm{Aut}(D)$, and in particular under the linear isometries of E. Associated with E_s is the *partial* (*Jordan*) *triple product* $\{\ \}: E \times E_s \times E \to E$, generalizing the

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triple product of JB^*-triples (and in particular the triple product $\{xyz\} = (xy^*z + zy^*x)/2$ in subtriples of $B(H)$). The linear isometries of E preserve the partial triple product and the skew hermitian bounded operators on E are derivations of the partial triple product.

In Theorem 2.6 below we prove that the symmetric part of a unital Jordan subalgebra $\mathfrak{A}$ of a JB^*-algebra E (and in particular, a unital subalgebra of $B(H)$) is simply $\mathfrak{A}_s = \mathfrak{A} \cap \mathfrak{A}^*$. Moreover, the partial triple product of $\mathfrak{A}$ is the restriction of the triple product of E. It follows that an isometry φ from a unital subalgebra $\mathfrak{A}$ of $B(H)$ onto a unital subalgebra $\mathscr{B}$ of $B(H)$ satisfies

$$\varphi(xy^*z + zy^*x) = \varphi(x)\,\varphi(y)^*\,\varphi(z) + \varphi(z)\,\varphi(y)^*\,\varphi(x)$$

for all $x, z \in \mathfrak{A}$, $y \in \mathfrak{A} \cap \mathfrak{A}^*$, and $\varphi(I)$ is a unitary in $\mathscr{B} \cap \mathscr{B}^*$. Thus $\psi(x) = \varphi(x)\,\varphi(I)^*$ is a isometry of $\mathfrak{A}$ onto $\mathscr{B}$, satisfying $\psi(I) = I$, which preserves also the Jordan product and the adjoint (when restricted to $\mathfrak{A} \cap \mathfrak{A}^*$). With $\mathfrak{A}$ and E as above, we also show in Theorem 2.14 that the Potapov Möbius transformations φ_a associated with $a \in D \cap \mathfrak{A} \cap \mathfrak{A}^*$ preserve not just $D \cap \mathfrak{A} \cap \mathfrak{A}^*$ but also $D \cap \mathfrak{A}$. This implies that every biholomorphic automorphism φ of $D \cap \mathfrak{A}$ is of the form $\varphi = \psi\varphi_a$, where ψ is a linear isometry of $\mathfrak{A}$, and $a = \varphi^{-1}(0)$.

Section 3 deals with isometries of nest algebras. By definition, a *nest* $\mathfrak{N}$ is a totally ordered set of projections on a Hilbert space H, containing 0 and 1. The associated *nest algebra* $\mathrm{alg}(\mathfrak{N})$ consists of all operators in $B(H)$ having $P(H)$ as an invariant subspace for every $P \in \mathfrak{N}$, i.e.,

$$\mathrm{alg}(\mathfrak{N}) := \{T \in B(H);\ TP = PTP,\ \forall P \in \mathfrak{N}\}.$$

Our main result in this section, Theorem 3.16, says that if $\mathfrak{N}$ and $\mathfrak{M}$ are nests of projections on H, which are complete (i.e., closed in the strong operator topology), then a linear isometry φ from $\mathrm{alg}(\mathfrak{N})$ onto $\mathrm{alg}(\mathfrak{M})$ is either of the form

$$\varphi(T) = UTU^*V, \qquad \forall T \in \mathrm{alg}(\mathfrak{N})$$

or of the form

$$\varphi(T) = U(JT^*J)\,U^*V, \qquad \forall T \in \mathrm{alg}(\mathfrak{N}),$$

where U, V are unitary operator, $V = \varphi(I)$ lies in the commutant $\mathfrak{M}' := \{T \in B(H);\ TP = PT,\ \forall P \in \mathfrak{M}\}$, and J is a fixed involution on H. Thus $P \mapsto \varphi(P)\,V^*$ is either an order isomorphism of $\mathfrak{N}$ onto $\mathfrak{M}$ (in the first case) or an order isomorphism of $\mathfrak{N}$ onto $\mathfrak{M}^\perp := \{I - P;\ P \in \mathfrak{M}\}$.

It follows that every linear isometry of $\mathrm{alg}(\mathfrak{N})$ onto $\mathrm{alg}(\mathfrak{M})$ can be extended uniquely to an isometry of $B(H)$.

The study of isometries yields quite easily the description of the hermitian operators (Theorem 3.10): Every bounded hermitian operator $h\colon \mathrm{alg}(\mathfrak{N}) \mapsto \mathrm{alg}(\mathfrak{N})$ is given by

$$h(T) = K_1 T - TK_2, \qquad T \in \mathrm{alg}(\mathfrak{N}),$$

where K_1, K_2 are self-adjoint elements of $\mathfrak{N}'$. In particular, h extends uniquely to a hermitian operator on $B(H)$.

Section 2 contains a survey of the Jordan theoretic and holomorphic tools needed in our proofs. The survey is a little more comprehensive than what is strictly needed, to make the exposition more self-contained.

The field of scalars is always assumed to be the complex numbers $\mathbb{C}$.

2. The Symmetric Part and Isometries of Jordan Subalgebras

We begin with a short survey on JB^*-triples and bounded symmetric domains. See the monographs [U1, U2, IS] for more details and proofs.

A complex Banach space E is a *JB^*-triple* if there exists a continuous sesquilinear form

$$E \times E \to B(E), \qquad (x, y) \mapsto D(x, y)$$

(where $B(E)$ denotes the space of bounded linear operators on E), such that

(i) the triple product $\{xyz\} := D(x, y)z$ is symmetric in x and z;

(ii) $D(x, x)$ is hermitian, i.e., $\|e^{itD(x,x)}\| = 1$ for all $t \in \mathbb{R}$, with spectrum in $[0, \infty)$;

(iii) $\|D(x, x)\| = \|x\|^2$;

(iv) the operators $\delta(x) = iD(x, x)$, $x \in E$, are *triple derivations*, i.e.,

$$\delta(x)(\{uvz\}) = \{\delta(x)\, u, v, z\} + \{u, \delta(x)\, v, z\} + \{u, v, \delta(x) z\}.$$

Let H be a Hilbert space. A closed subspace E of $B(H)$ which is closed under the map $x \to xx^*x$ is a JB^*-triple with respect to the triple product

$$\{xyz\} = (xy^*z + zy^*x)/2.$$

Such JB^*-triples are called *JC^*-triples* (or J^*-algebras by Harris [H1]).

An element e in a JB^*-triple E is called a *tripotent* if $\{eee\} = e$. In a JC^*-triple the tripotents are precisely the partial isometries. A tripotent e is *unitary* if $D(e, e) = I$.

A *JB^*-algebra* is a JB^*-triple E with a distinguished unitary tripotent e.

One defines the *Jordan product*

$$x \circ y = \{xey\}$$

and an involution

$$x^* = \{exe\}.$$

E is a Jordan algebra with respect to the Jordan product, namely

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$$

for all $x, y \in E$, where $x^2 = x \circ x$, and e is the unit of E. The involution satisfies, besides the usual properties, also

$$(x \circ y)^* = x^* \circ y^*.$$

The triple product can be expressed via the Jordan product and the involution by

$$\{xyz\} = x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*).$$

Every unital C^*-algebra is a JB^*-algebra. The symmetric matrices in $B(H)$ and spin factors are other examples of JB^*-algebras.

We survey now the approach to JB^*-triples via infinite dimensional holomorphy.

Let E be a complex Banach space and let D be its open unit ball. A function $h: D \to E$ is *holomorphic* (or analytic) in D if its Frechet derivative $h'(z)$ exists at every point $z \in D$. Equivalently, for each $z_0 \in D$ there is an open ball $B \subset D$ with center z_0 and a sequence $\{h_n\}_{n=0}^{\infty}$, where h_n is a homogeneous polynomial of degree n (i.e., the "diagonal" of a continuous, symmetric, n-linear map on E), so that

$$h(z) = \sum_{n=0}^{\infty} h_n(z - z_0), \qquad z \in B.$$

This is the Taylor series of h near z_0. A holomorphic function is also called a *holomorphic vector field.* A holomorphic function $h: D \to D$ is biholomorphic (or, a biholomorphic automorphism of D) if it is one-to-one, onto, and the inverse h^{-1} is holomorphic. We denote by $\mathrm{Aut}(D)$ the group of all *biholomorphic automorphisms* of D.

A fundamental result of W. Kaup is the following.

2.1. THEOREM [K1, K2]. *Let E be a complex Banach space with open unit ball D. Then $D = \mathrm{Aut}(D)(0)$ if and only if there exists on E a unique triple product, making it a JB^*-triple.*

We now proceed in describing the construction of the partial (Jordan) triple product in a general Banach space. For the details and the proofs see, besides the monographs [U1, U2, IS], also [V] and the survey [A].

Let E be a complex Banach space and let D be its open unit ball. Every holomorphic vector field $h: D \to E$ is *locally integrable*, namely the initial value problem

$$\frac{\partial}{\partial t}\varphi(t, z) = h(\varphi(t, z)), \qquad t \in J_z$$

$$\varphi(0, z) = z$$

has a unique solution in D for every $z \in D$. Here J_z is the maximal open interval containing $t_0 = 0$ for which the solution exists. The map $J_z \ni t \mapsto \varphi(t, z) \in D$ is called the *integral curve* of h through z. The holomorphic vector field h is *complete* if $J_z = \mathbb{R}$ for every $z \in D$. We write $\varphi_t(z) = \varphi(t, z)$. Then $\{\varphi_t\}_{t \in \mathbb{R}}$ is a norm-continuous one-parameter group of biholomorphic automorphisms of D called the *flow* associated with h. Clearly, $\varphi(t, z)$ is analytic in each variable. We shall frequently write $\exp(th)$ for φ_t.

One denotes by $\mathrm{aut}(D)$ the set of all complete holomorphic vector fields on D. A remarkable fact concerning the structure of $\mathrm{aut}(D)$ is that it admits a direct sum decomposition as a real Banach space, called the *Cartan decomposition*,

$$\mathrm{aut}(D) = \mathfrak{k} \oplus \mathfrak{p}.$$

$\mathfrak{k}$ is the set of all elements of $\mathrm{aut}(D)$ that can be extended to skew-Hermitian, bounded linear operators $A: E \to E$, i.e., $\{e^{tA}\}_{t \in \mathbb{R}}$ is a norm-continuous group of linear isometries of E. $\mathfrak{p}$ is the set of all members of $\mathrm{aut}(D)$ of the form

$$h_a(z) = a - Q_a(z), \qquad z \in D,$$

where Q_a is a quadratic homogeneous polynomial on E (depending on a) and $a = h_a(0)$ is a suitable point of E. In particular, the Taylor series about the origin of each $h \in \mathrm{aut}(D)$ contains only terms of degree $\leqslant 2$. In particular, each $h \in \mathrm{aut}(D)$ can be extended to a holomorphic map $h: E \to E$. If $h \in \mathrm{aut}(D)$, then $h \in \mathfrak{k}$ if and only if $h(-z) = -h(z)$, and $h \in \mathfrak{p}$ if and only if $h(-z) = h(z)$. One shows that $h_a \in \mathfrak{p}$ if and only if $h_{ia} \in \mathfrak{p}$ and that $Q_{ia} = -iQ_a$ and $Q_{a+b} = Q_a + Q_b$.

The *symmetric part* of E is defined to be

$$E_s := \mathrm{aut}(D)(0) = \{h(0); \qquad h \in \mathrm{aut}(D)\}.$$

The *symmetric part of D* is

$$D_s := D \cap E_s.$$

It is known that $E_s = E$ if and only if $\mathrm{Aut}(D)(0) = D$, i.e., if and only if E is a JB^*-triple.

If $a \in E_s$ one polarizes the homogeneous quadratic polynomial Q_a to get the symmetric bilinear map $E \times E \to E$ associated with a and denote it also by Q_a:

$$Q_a(a, y) = \tfrac{1}{2}\,(Q_a(x+y) - Q_a(x) - Q_a(y)).$$

The *partial (Jordan) triple product* is the map

$$\{\ \}: E \times E_s \times E \to E$$

defined via

$$\{xyz\} := Q_y(x, z), \qquad x, z \in E,\ y \in E_s.$$

For $x \in E$, $y \in E_s$ let $D(x, y) \in B(E)$ be the operator

$$D(x, y)z := \{xyz\}.$$

The following theorem summarizes some known properties of E_s and the partial triple product.

2.2. THEOREM. (i) *E_s is a closed, complex linear subspace of E with an open unit ball D_s.*

(ii) $D_s = \mathrm{Aut}(D)(0)$.

(iii) *D_s is invariant under all members of* $\mathrm{Aut}(D)$, *and* $\mathrm{Aut}(D)$ *acts on it transitively.*

(iv) *E_s is invariant under the triple product, $\{E_s, E_s, E_s\} \subseteq E_s$, and so E_s is a JB^*-triple.*

(v) *If ψ is a linear isometry of E and $h \in$* $\mathrm{aut}(D)$ *then $\psi \circ h \circ \psi^{-1} \in$* $\mathrm{aut}(D)$. *In particular, if $a \in E_s$ then $\psi \circ h_a \circ \psi^{-1} = h_{\psi(a)}$.*

(vi) *If ψ is a linear isometry of E then $\psi(E_s) = E_s$ and ψ is an automorphism of the partial triple product.*

Parts (i) and (ii) are not elementary, and they are proved in a slightly different form in [KU], see also [BKU]. Parts (iii)–(vi) are more elementary, see [U1, U2, KU, L] for proofs.

2.3. COROLLARY. *For $a, b \in E_s$, $iD(a, a)$ and $D(a, a) - D(b, a)$ are in k, hence they are derivations of the partial triple product.*

We conclude the survey with two known facts needed in the sequel. As before, E is a complex Banach space with open unit ball D. E' is the dual of E.

2.4. PROPOSITION. *Let F be a closed subspace of E and let $h \in \operatorname{aut}(D)$. Suppose that $h(D \cap F) \subseteq F$. Then*

$$h_{|D \cap F} \in \operatorname{aut}(D \cap F).$$

This follows by uniqueness of the solution of the initial value problem

$$\frac{\partial}{\partial t} \varphi(t, z) = h(\varphi(t, z))$$

$$\varphi(0, z) = z$$

for $z \in D \cap F$.

2.5. PROPOSITION. *Let $h: D \to E$ be holomorphic. Then $h \in \operatorname{aut}(D)$ if and only if h extends holomorphically to a neighborhood of $\bar{D}$, and for every $z \in E$, $f \in E'$ satisfying $\|z\| = 1 = \|f\| = f(z)$, we have*

$$\operatorname{Re} f(h(z)) = 0.$$

That is, the complete holomorphic vector fields are characterized by tangency to the unit sphere. See [S1, S2, U2, Lemma 4.4] for a proof.

Let E be a JB^*-algebra with a unit e. By a *Jordan subalgebra* $\mathfrak{A}$ of E we mean a closed linear subspace of E which is closed under the Jordan product $x \circ y = \{xey\}$. $\mathfrak{A}$ is not required to be closed under involution. Our main result in this section is the following.

2.6. THEOREM. *Let E be a JB^*-algebra with a unit e and let $\mathfrak{A}$ be a Jordan subalgebra of E which contains e. Then the symmetric part of $\mathfrak{A}$ is the maximal JB^*-subalgebra of E contained in $\mathfrak{A}$; that is,*

$$\mathfrak{A}_s = \mathfrak{A} \cap \mathfrak{A}^*.$$

The commutative analogue of this result, namely the characterization of the symmetric parts of function algebras, is known. See [BKU, IS].

Proof. Let $b \in \mathfrak{A} \cap \mathfrak{A}^*$. As the triple product $\{xby\}$ can be expressed via Jordan products of x, b^*, and y, we get $\{xby\} \in \mathfrak{A}$ for every $x, y \in \mathfrak{A}$. Let $h_b(x) = b - \{zbz\}$, $z \in \mathfrak{A}$. If D denotes the unit ball of E then

$h_b \in \operatorname{aut}(D)$. But since $h_b(D \cap \mathfrak{A}) \subseteq \mathfrak{A}$ we get by Proposition 2.4 that $h_{b|D \cap \mathfrak{A}} \in \operatorname{aut}(D \cap \mathfrak{A})$. Thus

$$b = h_b(0) \in \operatorname{aut}(D \cap \mathfrak{A})(0) = \mathfrak{A}_s.$$

This proves $\mathfrak{A} \cap \mathfrak{A}^* \subseteq \mathfrak{A}_s$.

Conversely, let $b \in \mathfrak{A}_s$. By definition this means that there exists $\tilde{h}_b \in \operatorname{aut} D \cap \mathfrak{A}$ of the form $\tilde{h}_b(z) = b - Q_b(z)$, where Q_b is a homogeneous quadratic polynomial on $\mathfrak{A}$. Let

$$S := \{f \in E';\ \|f\| = 1 = f(e)\}$$

be the state space of E. By Proposition 2.5 we get $0 = \operatorname{Re} f(\tilde{h}_b(e)), f \in S$, as well. Since $Q_{ib} = iQ_b$, we get

$$\overline{f(b)} = f(Q_b(e)), \qquad f \in S.$$

The same arguments applied to $h_b(z) = b - \{zbz\}$, which is complete in D, yield

$$\overline{f(b)} = f\{ebe\} = f(b^*), \qquad f \in S.$$

Comparing the last two identities we get

$$f(b^*) = f(Q_b(e)), \qquad f \in S.$$

Using the fact that e is unitary, and thus every element of E' can be written as a linear combination of members of S (this is due to the Jordan decomposition of self-adjoint elements if E', see [AS, Theorem 12.6]) we therefore get

$$b^* = Q_b(e) \in \mathfrak{A}.$$

Thus $b \in \mathfrak{A} \cap \mathfrak{A}^*$, and so $\mathfrak{A}_s \subseteq \mathfrak{A} \cap \mathfrak{A}^*$. This completes the proof. ∎

Our proof of Theorem 2.6 yields the following result.

2.7. COROLLARY. *Let E be a JB^*-algebra with a unit e, and let $\mathfrak{A}$ be a Jordan subalgebra containing e. Then the partial Jordan triple product of $\mathfrak{A}$ is the restriction to*

$$\mathfrak{A} \times \mathfrak{A}_s \times \mathfrak{A} = \mathfrak{A} \times (\mathfrak{A} \cap \mathfrak{A}^*) \times \mathfrak{A}$$

of the triple product of E. That is,

$$Q_b(z) = \{zbz\}, \qquad b \in \mathfrak{A} \cap \mathfrak{A}^*, z \in \mathfrak{A}.$$

Consequently, every quadratic complete holomorphic vector field $h_b(z) = z - Q_b(z)$ *on* $D \cap \mathfrak{A}$ *extends to a complete holomorphic vector field on* D.

2.8. COROLLARY. *Let* E_1, E_2 *be* JB^**-algebras with units* e_1, e_2*, respectively. Let* $\mathfrak{A}_1 \subseteq E_1$ *and* $\mathfrak{A}_2 \subseteq E_2$ *be Jordan subalgebras containing the units* e_1, e_2*, respectively, and let* $\varphi: \mathfrak{A}_1 \to \mathfrak{A}_2$ *be an isometry of* $\mathfrak{A}_1$ *onto* $\mathfrak{A}_2$*. Then*

$$\varphi(\mathfrak{A}_1 \cap \mathfrak{A}_2^*) = \mathfrak{A}_1 \cap \mathfrak{A}_2^*$$

$$\varphi\{xyz\} = \{\varphi(x)\,\varphi(y)\,\varphi(z)\}, \qquad x, z \in \mathfrak{A}_1,\ y \in \mathfrak{A}_1 \cap \mathfrak{A}_1^*$$

and $\varphi(e_1)$ *is a unitary tripotent in* $\mathfrak{A}_2 \cap \mathfrak{A}_2^*$*. Moreover, if* $\varphi(e_1) = e_2$ *then*

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y), \qquad x, y \in \mathfrak{A}_1$$

and

$$\varphi(x^*) = \varphi(x)^*, \qquad x \in \mathfrak{A}_1 \cap \mathfrak{A}_1^*.$$

In the important special case where $\mathfrak{A}$ is a norm closed unital subalgebra of $B(H)$ we get the following results.

2.9. COROLLARY. *Let* $\mathfrak{A}$ *be a norm closed subalgebra of* $B(H)$ *containing the identity operator* I.

(i) $\mathfrak{A}_s = \mathfrak{A} \cap \mathfrak{A}^*$*, i.e., the symmetric part of* $\mathfrak{A}$ *is the maximal* C^**-subalgebra of* $B(H)$ *contained in* $\mathfrak{A}$.

(ii) *The partial triple product in* $\mathfrak{A}$ *is given by*

$$\{xyz\} = (xy^*z + zy^*x)/2$$

for $x, z \in \mathfrak{A}$; $y \in \mathfrak{A} \cap \mathfrak{A}^*$.

2.10 COROLLARY. *Let* $\mathfrak{A} \subseteq B(H)$, $\mathscr{B} \subseteq B(K)$ *be unital norm closed subalgebras, and let* $\varphi: \mathfrak{A} \to \mathscr{B}$ *be a surjective linear isometry. Then*

(i) $\varphi(\mathfrak{A} \cap \mathfrak{A}^*) = \mathscr{B} \cap \mathscr{B}^*$.

(ii) $\varphi(xy^*z + zy^*x) = \varphi(x)\,\varphi(y)^*\,\varphi(z) + \varphi(z)\,\varphi(y)^*\,\varphi(x)$ *for every* $x, z \in \mathfrak{A}$*, and* $y \in \mathfrak{A} \cap \mathfrak{A}^*$*. In particular,* φ *maps partial isometries of* $\mathfrak{A} \cap \mathfrak{A}^*$ *to partial isometries of* $\mathscr{B} \cap \mathscr{B}^*$.

(iii) $\varphi(I) = V$ *is a unitary operator in* $B \cap B^*$*. If, moreover,* $\varphi(I) = I$*, then*

(iv) $\varphi(xy + yx) = \varphi(x)\,\varphi(y) + \varphi(y)\,\varphi(x)$: $x, y \in \mathfrak{A}$.

(v) $\varphi(x^*) = \varphi(x)^*$; $x \in \mathfrak{A} \cap \mathfrak{A}^*$.

A weaker version of this result was established by Harris [H2] who did not prove that $\varphi(I)$ is a unitary in $B \cap B^*$. Parts (iv) and (v) say that $\varphi_{|\mathfrak{A} \cap \mathfrak{A}^*}$ is a self-adjoint Jordan isomorphism of $\mathfrak{A} \cap \mathfrak{A}^*$ onto $\mathscr{B} \cap \mathscr{B}^*$. In particular φ maps projections to projections. It is well known and easy to prove that it is possible to characterize in terms of the Jordan product the notions of orthogonality of projections ($P \circ Q = 0$), commutativity of projections ($P \circ Q$ is a projection) and the order between them ($P \leqslant Q$ if and only if $P \circ Q = P$). Therefore one gets

2.11. COROLLARY. *Let $\mathfrak{A} \subseteq B(H)$, $\mathscr{B} \subseteq B(K)$ be unital subalgebras and let $\varphi\colon \mathfrak{A} \to \mathscr{B}$ be a surjective isometry satisfying $\varphi(I) = I$. Then*

(i) *φ restricts to an order isomorphism of the set of projections in $\mathfrak{A}$ onto the set of the projections in $\mathscr{B}$, which preserves commutativity and orthogonality;*

(ii) *for all $R, S, T \in \mathfrak{A}$ we have*

$$\varphi(RST + TSR) = \varphi(R)\,\varphi(S)\,\varphi(T) + \varphi(T)\,\varphi(S)\,\varphi(R).$$

Indeed, (i) follows from the previous discussion. The identity

$$STS + TST = (S+T)^3 - S^3 - T^3 - (TS^2 + S^2T) - (T^2S + ST^2)$$

together with the fact that φ preserves the Jordan product, gives

$$\varphi(TST + STS) = \varphi(T)\,\varphi(S)\,\varphi(T) + \varphi(S)\,\varphi(T)\,\varphi(S).$$

Replacing S by $-S$ and adding the resulting formula to the previous line yields

$$\varphi(TST) = \varphi(T)\,\varphi(S)\,\varphi(T).$$

From this the derived result (ii) follows by polarization.

A special but important case of Theorem 2.6 is worth an explicit formulation.

2.12. COROLLARY. *Let $\mathscr{L}$ be a set of projections on a Hilbert space H and let*

$$\mathfrak{A} = \operatorname{alg}(\mathscr{L}) := \{T \in B(H);\ TP = PTP,\ \forall P \in \mathscr{L}\}.$$

Then the symmetric part of $\mathfrak{A}$ is the commutant of $\mathscr{L}$:

$$\mathfrak{A}_s = \mathscr{L}' = \{T \in B(H);\ TP = PT,\ \forall P \in \mathscr{L}\}.$$

Remark. Let E and $\mathfrak{A}$ be as in Theorem 2.6 and let D be the open unit ball of E. Although we are interested here in the linear isometries of $\mathfrak{A}$, our analysis gives interesting information concerning the non-linear members of $\operatorname{Aut}(D \cap \mathfrak{A})$. We can show that if $a \in D \cap \mathfrak{A} \cap \mathfrak{A}^*$ then the corresponding Potapov–Möbius transformation φ_a (see [*Po*, IS]) maps $D \cap \mathfrak{A}$ onto itself, and that every $\varphi \in \operatorname{Aut}(D \cap \mathfrak{A})$ can be written uniquely as $\varphi = \psi\varphi_a$, where ψ is a surjective isometry of $\mathfrak{A}$ and $a = \varphi^{-1}(0)$.

3. Isometries of Nest Algebras

The main result of this section is the description of isometries of nest algebras. A family $\mathfrak{N}$ of projections in $B(H)$ is called a *nest* if it is totally ordered (where $P \leqslant Q$ if $P(H) \subseteq Q(H)$) and contains 0 and I; $\mathfrak{N}$ will be called a *complete* nest if, further, it is closed in the strong operator topology.

Given a complete nest and $0 \neq P \in \mathfrak{N}$ we define $P_- = \bigvee\{Q\colon Q \in \mathfrak{N}, Q < P\}$ (here $Q < P$ means $Q \leqslant P$ and $Q \neq P$). Clearly $P_- \in \mathfrak{N}$.

3.1. Definition. The *nest algebra* $\operatorname{alg} \mathfrak{N}$ associated with $\mathfrak{N}$ is defined to be the set of all operators in $B(H)$ which leave invariant the range space of each projection in $\mathfrak{N}$; i.e.,

$$\operatorname{alg} \mathfrak{N} = \{T \in B(H)\colon (I-P)\,TP = 0 \text{ for all } P \in \mathfrak{N}\}.$$

These algebras were introduced by Ringrose in [R1]. We shall frequently be interested in the operators of rank 1 in $\operatorname{alg} \mathfrak{N}$. We include here, for the sake of completeness, a few known results.

3.2. Lemma [R1, Lemma 3.3]. *Let $\mathfrak{N}$ be a complete nest of projections in $B(H)$ and let x and y be non-zero vectors in H. We write $x \otimes y$ for the operator $z \mapsto \langle z, x\rangle y$. Then $x \otimes y \in \operatorname{alg} \mathfrak{N}$ if and only if there is some $P \in \mathfrak{N}$ such that $P_- x = 0$ and $Py = y$.*

3.3. Lemma [R2, Lemma 2.3]

Let $\mathfrak{N}$ be a complete nest of projections in $B(H)$ and let $\mathfrak{A}_0$ be a subalgebra of $\operatorname{alg} \mathfrak{N}$ *which contains each operator of rank* 1 *in* $\operatorname{alg} \mathfrak{N}$. *Suppose $0 \neq T \in \mathfrak{A}_0$. Then T has rank 1 if and only if the following condition is satisfied: if $A, B \in \mathfrak{A}_0$ and $ATB = 0$, then at least one of AT, TB is zero.*

3.4. Lemma [E, Theorem 1]. *Let $\mathfrak{N}$ be a complete nest of projections and $T \in$* $\operatorname{alg} \mathfrak{N}$ *is an operator of finite rank, then T can be written as a finite sum of rank* 1 *operators of* $\operatorname{alg} \mathfrak{N}$.

We shall assume throughout this section that $\mathfrak{N}$ and $\mathfrak{M}$ are complete nests in $B(H)$. For a nest $\mathfrak{N}$ we write

$$\mathfrak{N}^{\perp} = \{I - P: P \in \mathfrak{N}\}$$

and refer to $\mathfrak{N}^{\perp}$ as the *complemented nest* of $\mathfrak{N}$.

For more details about nest algebras we refer the reader to [D].

Suppose now that φ is a linear isometry from a nest algebra $\operatorname{alg} \mathfrak{N}$ onto a nest algebra $\operatorname{alg} \mathfrak{M}$ (both algebras will be assumed to be subalgebras of $B(H)$). By Corollary 2.10 we know that φ maps $\operatorname{alg} \mathfrak{N} \cap (\operatorname{alg} \mathfrak{N})^*$ onto $\operatorname{alg} \mathfrak{M} \cap (\operatorname{alg} \mathfrak{M})^*$; as $\operatorname{alg} \mathfrak{N} \cap (\operatorname{alg} \mathfrak{N})^* = \mathfrak{N}' (= \{T \in B(H): TP = PT$ for every $P \in \mathfrak{N}\})$ we have $\varphi(\mathfrak{N}') = \mathfrak{M}'$. If we assume also that $\varphi(I) = I$ then it follows from Corollary 2.10 that φ preserves the Jordan product and the *-operation; i.e.,

$$\varphi(TS + ST) = \varphi(T)\,\varphi(S) + \varphi(S)\,\varphi(T)$$

$$\varphi(A^*) = \varphi(A^*)$$

for $A \in \mathfrak{N}'$, $S, T \in \operatorname{alg} \mathfrak{N}$. In this case, the restriction of φ to $\mathfrak{N}'$ preserves commutativity (see [K, Theorem 5]).

As the center of $\mathfrak{N}'$ is $\mathfrak{N}''$ (the von Neumann algebra generated by the projections in $\mathfrak{N}$), we have $\varphi(\mathfrak{N}'') = \mathfrak{M}''$. It is clear from the above that φ maps every projection in $\mathfrak{N}'$ into a projection in $\mathfrak{M}'$ and for two projections E, F in $\mathfrak{N}'$, $EF = 0$ if and only if $\varphi(E)\,\varphi(F) = 0$.

Our first goal is to show that a linear isometry φ from $\operatorname{alg} \mathfrak{N}$ onto $\operatorname{alg} \mathfrak{M}$ that satisfies $\varphi(I) = I$ induces an order isomorphism from $\mathfrak{N}$ onto either $\mathfrak{M}$ or $\mathfrak{M}^{\perp}$. This will be proved in Corollary 3.9. We first need a few lemmas.

For projections E, F in $\mathfrak{N}''$ we shall write $E \ll F$ if $EF = 0$ and $EB(H)F \subseteq \operatorname{alg} \mathfrak{N}$ (then we also have $F(\operatorname{alg} \mathfrak{N})E = \{0\}$). Similarly we define $\ll$ on $\mathfrak{M}''$.

3.5. LEMMA. *Suppose E and F are projections in $\mathfrak{N}''$ with $E \ll F$ and let φ be a linear isometry from $\operatorname{alg} \mathfrak{N}$ onto $\operatorname{alg} \mathfrak{M}$ with $\varphi(I) = I$. Then if $\varphi(E) \ll \varphi(F)$ we have $\varphi(EB(H)F) = \varphi(E)\,B(H)\,\varphi(F)$ and if $\varphi(F) \ll \varphi(E)$ then we have $\varphi(EB(H)F) = \varphi(F)\,B(H)\,\varphi(E)$.*

Proof. Suppose $\varphi(E) \ll \varphi(F)$. The proof for the other case is similar. Then $\varphi(F)(\operatorname{alg} \mathfrak{M})\,\varphi(E) = 0$.

It follows from Corollary 2.11 (ii) that

$$\begin{aligned}\varphi(EB(H)F) &\subseteq \varphi(E)(\operatorname{alg} \mathfrak{M})\,\varphi(F) + \varphi(F)(\operatorname{alg} \mathfrak{M})\,\varphi(E)\\ &= \varphi(E)(\operatorname{alg} \mathfrak{M})\,\varphi(F) = \varphi(E)\,B(H)\,\varphi(F).\end{aligned}$$

Applying a similar argument for φ^{-1} we have

$$\varphi^{-1}(\varphi(E)\,B(H)\,\varphi(F)) \subseteq EB(H)F$$

from which the result follows. ∎

3.6. LEMMA. *Suppose E and F are projections in $\mathfrak{N}''$ satisfying either $E \ll F$ or $F \ll E$ and φ is a linear isometry from* alg $\mathfrak{N}$ *onto* alg $\mathfrak{M}$ *with $\varphi(I) = I$. Then it is impossible to find non-zero projections E', E'', and F' in* M″ *satisfying*

(i) $E' \ll F' \ll E''$; *and*

(ii) $E' \leqslant \varphi(E)$, $E'' \leqslant \varphi(E)$, *and* $F' \leqslant \varphi(F)$.

Proof. We shall assume $E \ll F$. The proof for $F \ll E$ is similar. Suppose E', E'', and F' are projections in $\mathfrak{M}'$ satisfying (i) and (ii). Write $E_1 = \varphi^{-1}(E')$, $E_2 = \varphi^{-1}(E'')$, and $F_1 = \varphi^{-1}(F')$. Then $E_1 \ll F_1$ (as $E_1 \leqslant E$, $F_1 \leqslant F$) and $E_2 \ll F_1$. Using Lemma 3.5 we have

$$\varphi(E_1 B(H) F_1) = E' B(H) F'$$

and

$$\varphi(E_2 B(H) F_1) = E' B(H) E''.$$

If E_1, E_2, and F_1 are all non-zero projections then there are nonzero partial isometries u_1, u_2 such that $u_i \in E_i B(H) F_1$, $i = 1, 2$, and $u_1^* u_1 = u_2^* u_2$. But then $\varphi(u_1) \in E' B(H) F'$ and $\varphi(u_2) \in F' B(H) E''$. Hence $\|\varphi(u_1) + \varphi(u_2)\| = \max(\|\varphi(u_1\|, \|\varphi(u_2\|) = 1$. Since $\|u_1 + u_2\| \geqslant \sqrt{2}$ and φ is an isometry we arrive at a contradiction. ∎

3.7. LEMMA. *Suppose E and F are non-zero projections in $\mathfrak{N}''$ with $E \ll F$. Then either $\varphi(E) \ll \varphi(F)$ or $\varphi(F) \ll \varphi(E)$.*

Proof. Note first that if N and P are non-zero projections in $\mathfrak{M}''$ with $NP = 0$ and they do not satisfy $N \ll P$ then there are some non-zero subprojections $N_0 \leqslant N$ and $P_0 \leqslant P$ in $\mathfrak{M}''$ such that $N_0 \gg P_0$. For this, write

$$N' = \inf\{Q \in \mathfrak{M}\colon N(I - Q) = 0\}$$

and

$$P' = \sup\{Q \in \mathfrak{M}\colon QP = 0\}.$$

Since we do not have $N \ll P$ it follows that $N' > P'$. In fact, there is a projection $Q \in \mathfrak{M}$ satisfying $P' < N'$. Now write $N_0 = (N' - Q)N$ and $P_0 = (Q - P')P$ to get $N_0 \gg P_0$.

If E and F are as in the statement of the lemma and $\varphi(F) \ll \varphi(E)$ is not satisfied then there are non-zero subprojections $E_0 \leqslant E$ and $F_0 \leqslant F$ with $\varphi(E_0) \ll \varphi(F_0)$.

It follows that $\varphi(E) \ll \varphi(F_0)$. To see this suppose that this is not the case. Then there are non-zero subprojections $E_1 \leqslant E$ and $F_1 \leqslant F_0$ such that $\varphi(F_1) \ll \varphi(E_1)$; but then

$$\varphi(E_0) \ll \varphi(F_1) \ll \varphi(E_1)$$

contradicting Lemma 3.6.

If is left to show that $\varphi(E) \ll \varphi(F)$. If this is not so then there are non-zero subprojections $E_2 \leqslant E$, $F_2 \leqslant F$ such that $\varphi(F_2) \ll \varphi(E_2)$. But then $\varphi(F_2) \ll \varphi(E_2) \ll \varphi(F_0)$, again a contradiction to the result of Lemma 3.6. This contradiction completes the proof. ∎

3.8. Lemma. *For a projection $E \in \mathfrak{N}''$, E lies in $\mathfrak{N}$ if and only if $E \ll I - E$.*

Proof. It follows immediately from $\mathfrak{N} = lat \operatorname{alg} \mathfrak{N}$.

3.9. Corollary. *Let φ be a linear isometry from* alg $\mathfrak{N}$ *onto* alg $\mathfrak{M}$ *with $\varphi(I) = I$. Then one of the following two statements holds.*

(1) *φ induces an order isomorphism from $\mathfrak{N}$ onto $\mathfrak{M}$ and for every $T \in \operatorname{alg} \mathfrak{N}$ and projections P, Q in $\mathfrak{N}$, $\varphi(PTQ) = \varphi(P)\,\varphi(T)\,\varphi(Q)$.*

(2) *φ induces an order isomorphism from $\mathfrak{N}$ onto $\mathfrak{M}^-$ and for every $T \in \operatorname{alg} \mathfrak{N}$ and projections P, Q in $\mathfrak{N}$, $\varphi(PTQ) = \varphi(Q)\,\varphi(T)\,\varphi(P)$.*

Proof. If P is a projection in $\mathfrak{N}$ then, using Lemma 3.8, $P \ll I - P$; hence (Lemma 3.7) either $\varphi(P) \ll I - \varphi(P)$ or $I - \varphi(P) \ll \varphi(P)$. In the former case $\varphi(P) \in \mathfrak{M}$ and in the latter $\varphi(P) \in \mathfrak{M}^\perp$ (i.e., $I - \varphi(P) \in \mathfrak{M}$).

Suppose P and Q lie in $\mathfrak{N} \setminus \{0, 1\}$ and $\varphi(P) \in \mathfrak{M}$ while $\varphi(Q) \in \mathfrak{M}^\perp$. Since $PQ \neq 0$ and $(I-P)(I-Q) \neq 0$ we have $\varphi(P)\,\varphi(Q) \neq 0$ (i.e., $I - \varphi(Q) <$ $\varphi(P)$ as $I - \varphi(Q)$ and $\varphi(P)$ lie in $\mathfrak{M}$) and $(I - \varphi(Q))(I = \varphi(P)) \neq 0$ (i.e., $I - \varphi(Q) > \varphi(P)$). We arrive at a contradiction and this shows that either $\varphi(\mathfrak{N}) \subseteq \mathfrak{M}$ or $\varphi(\mathfrak{N}) \subseteq \mathfrak{M}^\perp$.

We now assume that φ maps $\mathfrak{N}$ into $\mathfrak{M}$ and complete the proof of statement (1). The proof of statement (2) is similar.

We have $\varphi(\mathfrak{N}) \subseteq \mathfrak{M}$ and can apply a similar analysis to φ^{-1} to get $\varphi(\mathfrak{N}) = \mathfrak{M}$. As φ is multiplicative on $\mathfrak{N}''$ (since it is a Jordan isomorphism and $\mathfrak{N}''$ is commutative), the map $P \mapsto \varphi(P)$ from $\mathfrak{N}$ onto $\mathfrak{M}$ is an order isomorphism. It follows that whenever $E \ll F$ (for projections E, F in $\mathfrak{N}''$) then $\varphi(E) \ll \varphi(F)$. The result now follows by applying Lemma 3.5 and Corollary 2.11 (ii). ∎

We now consider the algebra $\mathfrak{A}$ generated by $\mathfrak{N}'$ and $\bigcup \{PB(H)(I-P)\colon P \in \mathfrak{N}\}$ (and similarly, $\mathscr{B}$ generated by $\mathfrak{M}'$ and $\bigcup \{PB(H)(I-P)\colon P \in \mathfrak{M}\}$) and show that φ, restricted to $\mathfrak{A}$, is multiplicative and that $\mathfrak{A}$ contains all rank 1 operators in alg $\mathfrak{N}$. We shall then be able to conclude that φ maps the finite rank operators in alg $\mathfrak{N}$ into finite rank operators in alg $\mathfrak{M}$. This will allow us to extend φ to an isometry of alg $\mathfrak{N} + \mathscr{K}$ onto alg $\mathfrak{M} + \mathscr{K}$ (where $\mathscr{K}$ denotes the set of all compact operators in $B(H)$). We start with the following lemma.

3.10. Lemma. *Let φ be a linear isometry of* alg $\mathfrak{N}$ *onto* alg $\mathfrak{M}$ *mapping* $\mathfrak{N}$ *onto* $\mathfrak{M}$. *Let* $\mathfrak{U}$ *and* $\mathscr{B}$ *be the algebras defined above. Then*

(1) $\mathfrak{A}$ *is the linear subspace spanned by* $\bigcup \{PB(H)(I-P)\colon P \in \mathfrak{N}\}$ *and* $\mathfrak{N}'$ (*and* $\mathscr{B}$ *is the subspace spanned by* $\bigcup \{PB(H)(I-P)\colon P \in \mathfrak{M}\}$ *and* $\mathfrak{M}$).

(2) $\varphi(\mathfrak{A}) = \mathscr{B}$.

(3) $\mathfrak{A}$ [resp. $\mathscr{B}$] *contains all operators of rank* 1 *in* alg $\mathfrak{N}$ [*resp.* alg $\mathfrak{M}$].

Proof. In (1) and (3) we prove only the statements concerning $\mathfrak{A}$.

(1) It suffices to show that the linear subspace spanned by $\mathfrak{N}'$ and $\bigcup \{PB(H)(I-P)\colon P \in \mathfrak{N}\}$ is, in fact, an algebra. This is easy to check.

(2) follows immediately from Corollary 3.9 and the fact that $\varphi(\mathfrak{N}') = \mathfrak{M}'$.

(3) Let $x \otimes y$ be a rank 1 operator in alg $\mathfrak{N}$. Then (Lemma 3.2) there is some $P \in \mathfrak{N}$ such that $P_- x = 0$ and $Py = y$. Note that for every operators T, S in $B(H)$, $T(x \otimes y)S = S^* x \otimes Ty$. Write $x = x_0 + x_1$, where $x_0 = (P - P\ )x$ and $x_1 = (I-P)x$, and $y = y_0 + y_1$, where $y_1 = P\ \ y$ and $y_0 = (P - P\ )y$. Then

$$x \otimes y = x_1 \otimes y + x_0 \otimes y_0 + x_0 \otimes y_1$$

As $P(x_1 \otimes y)(I-P) = (I-P)x_1 \otimes P_y = x_1 \otimes y$, we have $x_1 \otimes y \in PB(H)(I-P) \subseteq \mathfrak{A}$. Similarly $x_0 \otimes y_1 \in P_- B(H)(I-P) \subseteq \mathfrak{A}$. Also $x_0 \otimes y_0 = (P - P_-)(x_0 \otimes y_0)(P - P\ ) \subseteq \mathfrak{N}'$ (as every $Q \in \mathfrak{N}$ satisfies either $Q \leqslant P$ or $Q \geqslant P$). This completes the proof. ∎

3.11. Lemma. *With* φ, $\mathfrak{A}$, *and* $\mathscr{B}$ *as in Lemma* 3.10, *the restriction of* φ *to* $\mathfrak{A}$ *is multiplicative provided* $\mathfrak{N} \neq \{0, 1\}$.

Proof. We first show that φ, restricted to $\mathfrak{N}'$, is multiplicative. For that fix $T \in B(H)$, $D_1, D_2 \in \mathfrak{N}'$, and $P \in \mathfrak{N} \setminus \{0, 1\}$. Using the fact that φ preserves the partial triple product (see Corollary 2.10 (ii)), we have

$$\begin{aligned}\varphi(PD_1D_2PT(I-P)) &= \varphi((PD_1)(D_2P)\,PT(I-P)) \\ &\quad + (PT(I-P))(D_2P)(PD_1)) \\ &= \varphi(PD_1)\,\varphi(D_2P)\,\varphi(PT(I-P)) \\ &\quad + \varphi(PT(I-P))\,\varphi(D_2P)\,\varphi(D_1P).\end{aligned}$$

Using the fact that $\varphi(PT(I-P)) = \varphi(P)\,\varphi(T)\,\varphi(I-P)$ (Corollary 3.9(a)) we get

$$\varphi(PD_1D_2PT(I-P)) = \varphi(PD_1)\,\varphi(D_2P)\,\varphi(PT(I-P)). \tag{1}$$

But also,

$$\begin{aligned}\varphi(PD_1D_2PT(I-P)) &= \varphi((PD_1D_2P)(PT(I-P)) \\ &\quad + (PT(I-P))(PD_1D_2P)) \\ &= \varphi(PD_1D_2P)\,\varphi(PT(I-P)) \\ &\quad + \varphi(PT(I-P))\,\varphi(PD_1D_2P) \\ &= \varphi(D_1D_2P)\,\varphi(PT(I-P)).\end{aligned}$$

Combining this with (1) we have

$$(\varphi(PD_1)\,\varphi(D_2P) - \varphi(D_1D_2P))\,\varphi(PT(I-P)) = 0$$

and consequently

$$(\varphi(P)\,\varphi(D_1)\,\varphi(D_2) - \varphi(P)\,\varphi(D_1D_2))\,B(H)\,\varphi(I-P) = 0 \tag{2}$$

As $I-P \neq 0$ and $B(H)$ is a factor,

$$\varphi(P)(\varphi(D_1)\,\varphi(D_2) - \varphi(D_1D_2)) = 0. \tag{3}$$

If we now consider $\varphi(PT(I-P)\,D_1D_2(I-P))$, instead of $\varphi(PD_1D_2PT(I-P))$ and perform a similar computation, we obtain

$$\begin{aligned}&\varphi(PT(I-P)\,D_1D_2(I-P)) \\ &\quad = \varphi(PT(I-P))\,\varphi((I-P)D_1)\,\varphi((I-P)D_1)\,\varphi((I-P)D_2)\end{aligned} \tag{1$'$}$$

instead of (1). In place of (2) we have

$$\varphi(P)\,B(H)\,\varphi(I-P)(\varphi(D_1)\,\varphi(D_2) - \varphi(D_1D_2)) = 0 \tag{2$'$}$$

and in place of (3),

$$\varphi(I-P)(\varphi(D_1)\,\varphi(D_2) - \varphi(D_1D_2)) = 0. \tag{3$'$}$$

As $\varphi(P)+\varphi(I-P)=\varphi(I)=I$, $\varphi(D_1 D_2)=\varphi(D_1 D_2) = \varphi(D_1)\varphi(D_2)$. This completes the proof of the multiplicativity of φ on $\mathfrak{N}'$.

In fact, we proved more. Notice that (1) and (1′) imply (when we set $D_2=I$, $D_1=D$) that

$$\varphi(DPT(I-P))=\varphi(D)\,\varphi(PT(I-P))$$

and
$$\varphi(PT(I-P)D)=\varphi(PT(I-P))\,\varphi(D)$$

for every $T\in B(H)$, $D\in\mathfrak{N}'$. It is left, therefore, only to show that for $P, Q\in\mathfrak{N}$, $T, S\in B(H)$ we have

$$\varphi(PT(I-P)\,QS(I-Q))=\varphi(PT(I-P))\,\varphi(QS(I-Q)).$$

If $Q\leqslant P$ then $(I-P)Q=\varphi(I-P)\,\varphi(Q)=0$ and equality holds. Assume now that $P\leqslant Q$. Set $R=PT(I-P)$ and $C=QS(I-Q)$ and notice that $CR=0=\varphi(C)\,\varphi(R)$. We have

$$\begin{aligned}\varphi(PT(I-P)\,QS(I-Q))&=\varphi(RC)=\varphi(RC+CR)\\&=\varphi(R)\,\varphi(C)+\varphi(C)\,\varphi(R)\\&=\varphi(R)\,\varphi(C)=\varphi(PT(I-P))\,\varphi(QS(I-Q)).\quad\blacksquare\end{aligned}$$

3.12. COROLLARY. *$\varphi(\mathfrak{J}\cap\operatorname{alg}\mathfrak{N})=\mathfrak{J}\cap\operatorname{alg}\mathfrak{M}$, where $\mathfrak{J}$ denotes the set of all finite rank operators in $B(H)$ and φ is as in Lemma* 3.10.

Proof. Suppose $\mathfrak{N}\neq\{0,1\}$. Then φ is multiplicative on $\mathfrak{A}$ (Lemma 3.11) and $\mathfrak{A}$ contains all rank 1 operators in $\operatorname{alg}\mathfrak{N}$ (Lemma 3.10(3)). It now follows from Lemma 3.3 that if T is a rank 1 operator in $\operatorname{alg}\mathfrak{N}$ then $\varphi(T)$ is a rank 1 operator in $\operatorname{alg}\mathfrak{M}$. Using Lemma 3.4 this proves that $\varphi(\mathfrak{J}\cap\operatorname{alg}\mathfrak{N})\subseteq\mathfrak{J}\cap\operatorname{alg}\mathfrak{M}$ when $\mathfrak{N}\neq\{0,1\}$. For $\mathfrak{N}=\{0,1\}$ we know from the results of [K, Corollary 11] that φ is either a *-isomorphism or a *-anti-isomorphism of $B(H)$ ($=\operatorname{alg}\{0,1\}$). It follows that φ maps finite rank operators into finite rank operators. This proves that we always have $\varphi(\mathfrak{J}\cap\operatorname{alg}\mathfrak{N})\subseteq\mathfrak{J}\cap\operatorname{alg}\mathfrak{M}$ and since this holds also for φ^{-1} we have equality. $\blacksquare$

Since $\mathfrak{J}\cap\operatorname{alg}\mathfrak{N}$ is σ-weakly dense in $\operatorname{alg}\mathfrak{N}$ [FAM, Appendix Corollary 2] and $\operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*$ is σ-weakly dense in $B(H)$, we have the following:

3.13. LEMMA. *The subspace $(\mathfrak{J}\cap\operatorname{alg}\mathfrak{N})+(\mathfrak{J}\cap\operatorname{alg}\mathfrak{N})^*$ is norm dense in the algebra $\mathscr{K}$ of all compact operators in $B(H)$.*

3.14 LEMMA. *Suppose* φ *is as in Lemma* 3.10. *Then there is an extension of* φ *to an isometry,* $\tilde{\varphi}$, *of the norm closure of* $\operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*$ *onto the norm closure of* $\operatorname{alg}\mathfrak{M}+(\operatorname{alg}\mathfrak{M})^*$ *mapping* $\mathscr{K}$ *(the algebra of all compact operators in* $B(H)$*) onto itself.*

Proof. Using [Ar1, Proposition 1.2.8] (see also [P, Proposition 2.12]) we can extend φ to a positive map $\tilde{\varphi}$ from $\operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*$ onto $\operatorname{alg}\mathfrak{M}+(\operatorname{alg}\mathfrak{M})^*$ defined by

$$\tilde{\varphi}(T+S^*)=\varphi(T)+\varphi(S)^*, \qquad T, S\in \operatorname{alg}\mathfrak{N}.$$

Similarly we get a positive extension φ' of φ^{-1} with

$$\varphi'(T+S^*)=\varphi^{-1}(T)+(\varphi^{-1}(S))^*, \qquad T, S\in \operatorname{alg}\mathfrak{M}.$$

Clearly $\varphi'=\tilde{\varphi}^{-1}$. Since both $\tilde{\varphi}^{-1}$ and $\tilde{\varphi}$ are positive and unital, $\|\tilde{\varphi}\|\leqslant 1$ and $\|\tilde{\varphi}^{-1}\|\leqslant 1$ (see [P, Corollary 2.8]). Hence $\tilde{\varphi}$ is an isometry and can be extended to the norm closure of $\operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*$ mapping $\mathscr{K}$ onto itself.

3.15. LEMMA. *Let* φ *be a linear isometry of* $\operatorname{alg}\mathfrak{N}$ *onto* $\operatorname{alg}\mathfrak{M}$ *mapping* $\mathfrak{N}$ *onto* $\mathfrak{M}$ *and assume* $\mathfrak{N}\neq\{0,1\}$. *Let* $\tilde{\varphi}$ *be the extension of* φ *to an isometry of the norm closure of* $\operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*$ *onto the norm closure of* $\operatorname{alg}\mathfrak{M}+(\operatorname{alg}\mathfrak{M})^*$ *as in Lemma* 3.14. *Then there is a unitary operator* $U\in B(H)$ *such that*

$$\tilde{\varphi}(T)=UTU^*, \qquad T\in \operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*.$$

Proof. Since $\tilde{\varphi}$, restricted to $\mathscr{K}$, is an isometry of $\mathscr{K}$ onto itself (Lemma 3.14) it follows from Theorem 6.4 in [St] that $\tilde{\varphi}$, restricted to $\mathscr{K}$, is either a *-isomorphism or a *-anti-isomorphism. Since $\tilde{\varphi}$, restricted to $\mathfrak{J}\cap\operatorname{alg}\mathfrak{N}$ is multiplicative (Lemma 3.11 and Lemma 3.10(3)), $\tilde{\varphi}$ is multiplicative on $\mathscr{K}$. Therefore (see [Ar2, Corollary 3, p. 20]) there is some unitary operator $U\in B(H)$ satisfying $\tilde{\varphi}(T)=UTU^*$ for every $T\in\mathscr{K}$. We now use an argument of [HP].

For every x, y in H and T in the norm closure of $\operatorname{alg}\mathfrak{N}+(\operatorname{alg}\mathfrak{N})^*$ we set

$$f(T)=\langle\tilde{\varphi}(T)\,x, y\rangle$$
$$g(T)=\langle UTU^*x, y\rangle.$$

Since $|f(T)|\leqslant\|T\|\;\|x\|\;\|y\|$, $\|f\|\leqslant\|x\|\;\|y\|$. For $T=U^*x\otimes U^*y$, $f(T)=g(T)=\langle U(U^*x\otimes U^*y)\,U^*x, y\rangle=\|U^*x\|^2\,\|U^*y\|^2=\|x\|\;\|y\|\;\|T\|$. As $U^*x\otimes U^*y$ lies in $\mathscr{K}$ we have $\|f\|=\|f\mid\mathscr{K}\|=\|x\|\;\|y\|$. Proposition 10.4.1 of [KR] now implies that f is σ-weakly continuous. As f and g agree on

$\mathscr{K}$ and both are σ-weakly continuous we have $f = g$. Since this holds for all x, y in H,

$$\tilde{\varphi}(T) = UTU^*, \qquad T \in \operatorname{alg} \mathfrak{N} + (\operatorname{alg} \mathfrak{N})^*. \quad \blacksquare$$

For the following fix an involution J of H, i.e., an isometric conjugate-linear mapping J of H onto H such that $J^2 = I$. Then it is easy to check that the map $T \mapsto JT^*J$ is a $*$-anti-isomorphism of $B(H)$ onto itself.

We now turn to the main result of this section.

3.16. THEOREM. *Let $\mathfrak{N}$ and $\mathfrak{M}$ be complete nests in $B(H)$ and φ be a linear isometry from* $\operatorname{alg} \mathfrak{N}$ *onto* $\operatorname{alg} \mathfrak{M}$. *Then there are unitary operators U in $B(H)$ and $V = \varphi(I)$ in $\mathfrak{M}'$ such that one of the following two cases holds.*

Case (1). $\varphi(T) = UTU^*V$ for every $T \in \operatorname{alg} \mathfrak{N}$ and the map $N \mapsto UNU^* = \varphi(N)V^*$ is an order isomorphism on $\mathfrak{N}$ onto $\mathfrak{M}$.

Case (2). $\varphi(T) = UJT^*JU^*V$ for every $T \in \operatorname{alg} \mathfrak{N}$ (where J is a fixed involution on H) and the map $N \mapsto UJNJU^* = \varphi(N)V^*$ is an order isomorphism on $\mathfrak{N}$ onto $\mathfrak{M}^\perp = \{I - P \colon P \in \mathfrak{M}\}$.

Proof. First suppose that $\mathfrak{N} = \{0, 1\}$. Then $\mathfrak{M} = \{0, 1\}$ and $\operatorname{alg} \mathfrak{N} = \operatorname{alg} \mathfrak{M} = B(H)$. By [K, Corollary 11 and Theorem 7] the map $T \mapsto \varphi(T)\,\varphi(I)^*$ is either a $*$-isomorphism or a $*$-anti-isomorphism of $B(H)$; hence either Case (1) or Case (2) holds.

Now suppose $\mathfrak{N} \neq \{0, 1\}$. As φ is a linear isometry of $\mathfrak{N}'$ onto $\mathfrak{M}'$ (Corollary 2.19(i) and Corollary 2.12) we know that $\varphi(I)$ is a unitary operator in $\mathfrak{M}'$, to be denoted by V(see [K, Theorem 7]). Write $\psi(T) = \varphi(T)V^*$. Then ψ is a isometry from $\operatorname{alg} \mathfrak{N}$ onto $\operatorname{alg} \mathfrak{M}$ with $\psi(I) = I$. Using Corollary 3.9 we see that either $\psi(\mathfrak{N}) = \mathfrak{M}$ or $\psi(\mathfrak{N}) = \mathfrak{M}^\perp$.

If $\psi(\mathfrak{N}) = \mathfrak{M}$, then Lemma 3.15 shows that Case (1) holds. If $\psi(\mathfrak{N}) = \mathfrak{M}^\perp$ and J is a fixed involution on H, define

$$\psi_0(T) = \psi(JT^*J).$$

Then ψ_0 is a linear isometry from $\operatorname{alg}((J\mathfrak{N}J)^\perp)$ (where $J\mathfrak{N}J = \{JNJ \colon N \in \mathfrak{N}\}$) onto $\operatorname{alg} \mathfrak{M}$ and it maps $(JNJ)^\perp$ onto $\mathfrak{M}$. Applying Lemma 3.15 to ψ_0 we complete the proof in Case (2). $\blacksquare$

3.17. COROLLARY. *If $\mathfrak{N}$ is not order isomorphic to $\mathfrak{N}^\perp$ then every isometry φ of* $\operatorname{alg} \mathfrak{N}$ *onto itself is of the form $\varphi(T) = UTU^*V$, $T \in \operatorname{alg} \mathfrak{N}$, where U is a unitary operator in $B(H)$ and V is a unitary operator in $\mathfrak{M}'$.*

Proof. The corollary follows immediately from the theorem as Case (2) cannot hold.

The following corollary follows immediately from the theorem.

3.18. COROLLARY. *Every linear isometry of* alg $\mathfrak{N}$ *onto* alg $\mathfrak{M}$ *can be extended to a linear isometry of* $B(H)$.

We shall now use Theorem 3.16 to study the hermitian operators on alg $\mathfrak{N}$ for a complete nest $\mathfrak{N}$. Recall that a bounded linear operator $\gamma\colon \operatorname{alg}\mathfrak{N} \to \operatorname{alg}\mathfrak{N}$ is called *hermitian* if for every $t \in \mathbb{R}$, $\|\exp(it\gamma)\| = 1$ (see [BD, Definition 5.1, Lemma 5.2]. Clearly, in this case, $\exp(it\gamma)$ is an isometry on alg $\mathfrak{N}$ for every $t \in \mathbb{R}$.

3.19. THEOREM. *If* $\gamma\colon \operatorname{alg}\mathfrak{N} \to \operatorname{alg}\mathfrak{N}$ *is an hermitian operator then there are self-adjoint operators* K *and* S *in* $\mathfrak{N}'$ *such that* $\gamma(T) = ST - TK$ *for every* $T \in \operatorname{alg}\mathfrak{N}$.

Proof. The proof starts with an argument similar to the one used in [Si, Remark 3.5]. For any state f on $B(H)$, $\tilde{f}(\beta) = f(\beta(I))$ defines a continuous linear functional $\tilde{f}$ on $B(\operatorname{alg}\mathfrak{N})$ (the bounded operators on alg $\mathfrak{N}$) of norm 1 satisfying $\tilde{f}(I) = 1$. Since γ is hermitian, $\tilde{f}(\gamma) \in \mathbb{R}$ for every state f [BD, Lemma 5.2]. Hence $\gamma(I) = \gamma(I)^*$ and, since $\gamma(I)$ lies in alg $\mathfrak{N}$, $\gamma(I) \in \mathfrak{N}'$. The operator defined on alg $\mathfrak{N}$ by multiplication by $\gamma(I)$ is a hermitian operator on alg $\mathfrak{N}$. By substracting this operator from γ we obtain an hermitian operator γ_0 (as the set of all hermitian operators on a Banach algebra is a real linear space) satisfying $\gamma_0(I) = 0$. Now write

$$\varphi_t = \exp(it\gamma_0), \qquad t \in \mathbb{R}.$$

Then, for every $t \in \mathbb{R}$, φ_t is an isomeltry of alg $\mathfrak{N}$ onto itself with $\varphi_t(I) = \exp(it\gamma_0)(I) = I$ (as $\gamma_0(I) = 0$). Hence, for every $t \in \mathbb{R}$, φ_t is either a *-isomorphism or a *-anti-isomorphism (Theorem 3.16). Let B be the set of all $t \in \mathbb{R}$ such that φ_t is an isomorphism. Since $t \mapsto \varphi_t$ is a norm continuous one parameter group of isometries of alg $\mathfrak{N}$, the set B is closed and so is the set $\mathbb{R} \setminus B$. Since $\mathbb{R}$ is connected and $0 \in B$, $\mathbb{R} = B$; i.e., every φ_t is a *-isomorphism of alg $\mathfrak{N}$.

Hence γ_0 is a derivation on alg $\mathfrak{N}$. Therefore [Ch, Corollary 3.11] there is some operator $K \in \operatorname{alg}\mathfrak{N}$ such that $\gamma_0(T) = KT - TK$, $T \in \operatorname{alg}\mathfrak{N}$.

We now have,

$$\exp(it\gamma_0)(T) = (\exp itK)\, T(\exp(-itK)), \qquad T \in \operatorname{alg}\mathfrak{N},\ t \in \mathbb{R}.$$

But $\exp(it\gamma_0)$ is a multiplicative isometry on alg $\mathfrak{N}$ that maps I into I; hence there are unitary operators $U_t \in B(H)$ ($t \in \mathbb{R}$) such that

$$U_t T U_t^* = \exp(itK)\, T(\exp(-itK)), \qquad t \in \mathbb{R},\ T \in \operatorname{alg}\mathfrak{N}.$$

This shows that $\exp(itK)\,\mathfrak{N}' \exp(itK) \subseteq \mathfrak{N}'$ and, consequently, $\gamma_0(\mathfrak{N}') \subseteq \mathfrak{N}'$. (This fact can also be seen using Theorem 2.2.) In particular $KP - PK \in \mathfrak{N}'$

for every $P \in \mathfrak{N}$ and, therefore $PKP - PK = PK - PKP$. As $K \in \operatorname{alg} \mathfrak{N}$, $KP - PKP = 0$ and we have

$$PK = PKP = KP, \qquad P \in \mathfrak{N}.$$

Hence $K \in \mathfrak{N}'$. Write $K = K_1 + iK_2$, where K_i, $i = 1, 2$, are self-adjoint operators in $\mathfrak{N}'$ and let $\gamma_i(T) = K_i T - TK_i$, for $T \in \operatorname{alg} \mathfrak{N}$. Then $\gamma_0 = \gamma_1 + i\gamma_2$ and γ_1 is hermitian. Hence both γ_2 and $i\gamma_2$ are hermitian which implies $\gamma_2 = 0$ and $\gamma_0(T) = K_1 T - TK_1$, $T \in \operatorname{alg} \mathfrak{N}$. Finally, $\gamma(T) = \gamma_0(T) + \gamma(I) T = (K_1 + \gamma(I)) T - TK_1$, $T \in \operatorname{alg} \mathfrak{N}$. ▮

3.20. COROLLARY. *Every complete holomorphic vector field on the unit ball of* alg $\mathfrak{N}$ *(for a complete nest* $\mathfrak{N}$*) extends to a complete holomorphic vector field on the unit ball of* $B(H)$.

Proof. Let D be the unit ball of $B(H)$ and recall (Section 2) that

$$\operatorname{aut}(D \cap \operatorname{alg} \mathfrak{N}) = \mathscr{k} \oplus \mathscr{p}.$$

Corollary 2.7 shows that the members of $\mathscr{p}$ extend to complete holomorphic vector fields on D and Theorem 3.19 provides an extension for every member of $\mathscr{k}$, thus completing the proof. ▮

Note added in proof. R. Moore and T. Trent have independently proved Theorem 3.16 using different methods. (See *J. Functional Analysis* **86** (1989), 180–210).

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